Resit exam — Ordinary Differential Equations (WIGDV-07)

Monday 2 March 2015, 18.30h–21.30h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Question 1 (10 points)

Solve the following initial value problem:

$$x^2 \frac{dy}{dx} = 3(x^2 + y^2) \arctan\left(\frac{y}{x}\right) + xy, \qquad y(1) = 1.$$

What is the largest interval on which the solution exists?

Question 2 (10 points)

Solve the following Bernoulli equation:

$$\frac{dy}{dx} + y = \frac{x}{y^2}.$$

Question 3 (10 points)

Use an integrating factor of the form $M(x, y) = \phi(x)$ to solve the following equation:

$$y\,dx + (x^2 y - x)\,dy = 0.$$

Question 4 (15 points)

Compute e^{At} for the following 3×3 matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & -2 & 3 \end{bmatrix}.$$

Question 5 (5 + 10 + 5 points)

Let $C([0,b])$ be the space of continuous functions $y : [0,b] \to \mathbb{C}$ which is equipped with the norm

$$\|y\| = \sup_{x\in[0,b]} |y(x)|.$$

Consider the integral operator

$$T : C([0,b]) \to C([0,b]), \qquad (Ty)(x) = x + \lambda \int_0^x (x-t)y(t)\,dt,$$

where $\lambda \in \mathbb{C}$ is a parameter.

(a) Prove that if $Ty = y$, then y satisfies the initial value problem

$$y'' = \lambda y, \qquad y(0) = 0, \qquad y'(0) = 1.$$

(b) Prove that

$$\|Ty - Tz\| \le \tfrac{1}{2}|\lambda| b^2 \cdot \|y - z\|, \qquad \forall\, y, z \in C([0,b]).$$

(c) Let $y_0(x) = x$ and define the sequence $y_{n+1} = Ty_n$. Prove by induction that

$$y_n(x) = \sum_{k=0}^{n} \frac{\lambda^k x^{2k+1}}{(2k+1)!}, \qquad \forall n \in \mathbb{N} \cup \{0\}.$$

Question 6 (2 + 3 + 5 points)

Consider the following second order equation:

$$x^2 u'' - 2xu' + 2u = x^3 \sin x.$$

(a) Why is this equation called *linear*?

(b) Solve the homogeneous equation by substituting $u(x) = x^\lambda$.

(c) Compute a particular solution of the form $u_p(x) = Ax^m \sin x$.

Question 7 (15 points)

Compute all real eigenvalues λ and corresponding eigenfunctions u for the following boundary value problem:

$$u'' + \lambda u = 0, \qquad u'(0) = 0, \qquad u'(\pi) = 0.$$

Hint: consider the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ separately.

End of test (90 points)

Solution question 1 (10 points)

- First we rewrite the differential equation as

$$\frac{dy}{dx} = 3\left(1 + \frac{y^2}{x^2}\right) \arctan\left(\frac{y}{x}\right) + \frac{y}{x}.$$

  The variable $u = y/x$ satisfies a differential equation with separated variables:

$$\frac{du}{dx} = \frac{3(1+u^2)\arctan(u)}{x} \qquad \Rightarrow \qquad \int \frac{1}{(1+u^2)\arctan(u)}\,du = \int \frac{3}{x}\,dx.$$

  (2 points)

- Working out the integrals gives

$$\log|\arctan(u)| = 3\log|x| + C \quad \Rightarrow \quad \arctan(u) = Kx^3 \quad \Rightarrow \quad u = \tan(Kx^3),$$

  where $K = \pm e^C$. Hence, the general solution of the differential equation is given by

$$y = x\tan(Kx^3).$$

  (4 points)

- The initial condition $y(1) = 1$ implies that $1 = \tan(K)$ so that $K = \pi/4$.
  (2 points)

- The solution exists on the open interval $(-\sqrt[3]{2}, \sqrt[3]{2})$.
  (2 points)

Solution question 2 (10 points)

- The exponent of the nonlinear term is $\alpha = -2$. Therefore we introduce the new variable $z = y^{1-\alpha} = y^3$ which satisfies a linear differential equation:

$$z' + 3z = 3x.$$

  (3 points)

- Multiplying the equation with the integrating factor $\phi(x) = e^{3x}$ gives

$$e^{3x}z' + 3e^{3x}z = 3xe^{3x} \quad \Rightarrow \quad \frac{d}{dx}\left[e^{3x}z\right] = 3xe^{3x} \quad \Rightarrow \quad z = x - \tfrac{1}{3} + Ce^{-3x}.$$

  (5 points)

- Hence, the solution of the Bernoulli equation is given by

$$y = \sqrt[3]{x - \tfrac{1}{3} + Ce^{-3x}}.$$

  (2 points)

Remark. The linear differential equation for z can also be solved by first solving the homogeneous equation and then applying variation of constants to find a particular solution.

Solution question 3 (10 points)

- Define the functions

$$g(x,y) = y\phi(x) \quad \text{and} \quad h(x,y) = (x^2y - x)\phi(x).$$

The equation becomes exact if and only if

$$g_y = h_x \quad \Leftrightarrow \quad \phi(x) = (2xy-1)\phi(x) + (x^2y-x)\phi'(x) \quad \Leftrightarrow \quad \phi'(x) = -\frac{2}{x}\phi(x).$$

An obvious solution is $\phi(x) = 1/x^2$.
(4 points)

- We have

$$g(x,y) = \frac{y}{x^2} \quad \text{and} \quad h(x,y) = y - \frac{1}{x}.$$

Next, we define a potential function by

$$F(x,y) = \int g(x,y)\,dx + \psi(y) = -\frac{y}{x} + \psi(y).$$

By construction the equality $F_x = g$ holds. In addition, the equality $F_y = h$ holds if and only if $\psi'(y) = y$. We can take $\psi(y) = \frac{1}{2}y^2$.
(4 points)

- The solution of the differential equation is given by the implicit equation

$$F(x,y) = C \quad \Leftrightarrow \quad -\frac{y}{x} + \tfrac{1}{2}y^2 = C,$$

where C is an arbitrary constant.
(2 points)

Solution question 4 (15 points)

- The characteristic polynomial of the matrix A is given by

$$\det(A - \lambda I) = (2-\lambda)^3.$$

Hence, $\lambda = 2$ is the only eigenvalue of A with multiplicity three.
(2 points)

- Straightforward computations show that

$$A - I = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad (A-I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, the first two generalized eigenspaces of A are given by

$$E_\lambda^1 = \text{Nul}(A-I) \;\; = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\},$$

$$E_\lambda^3 = \text{Nul}(A-I)^2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Hence, the associated dot diagram is given by

$$\left.\begin{aligned} r_1 &= \dim E^1_\lambda = 2 \\ r_2 &= \dim E^2_\lambda - \dim E^1_\lambda = 1 \end{aligned}\right\} \Rightarrow \begin{matrix} \bullet & \bullet \\ \bullet & \end{matrix}$$

This means that we have one cycle of length two and one cycle of length one. **(4 points)**

- The 1-cycle is just a vector $\mathbf{v} \in E^1_\lambda$. For example, we can take

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

The 2-cycle is given by $\{(A - I)\mathbf{w}, \mathbf{w}\}$ where $\mathbf{w} \in E^2_\lambda \setminus E^1_\lambda$. For example, we can take

$$\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad (A - I)\mathbf{w} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

(2 points)

- If we choose to list the 1-cycle first, then the Jordan canonical form is $A = QJQ^{-1}$ with

$$Q = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix},$$

(2 points)

- The inverse of the matrix Q is given by

$$Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix},$$

(3 points)

- Hence, we obtain

$$e^{At} = Qe^{Jt}Q^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 1-t & 2t & -t \\ 0 & 1 & 0 \\ t & -2t & 1+t \end{bmatrix}.$$

(2 points)

Remark. This question can be solved without using the Jordan canonical form. We can write $A = D + N$ where

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

Clearly, the matrices D and N commute, that is $DN = ND$. Therefore, we can apply the rule

$$e^{At} = e^{(D+N)t} = e^{Dt}e^{Nt}.$$

Note that the matrix N is nilpotent:

$$N^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In particular, it follows that $N^k = 0$ for all $k \geq 2$. We have

$$e^{Dt} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \quad \text{and} \quad e^{Nt} = I + Nt = \begin{bmatrix} 1-t & 2t & -t \\ 0 & 1 & 0 \\ t & -2t & 1+t \end{bmatrix}.$$

Finally, we obtain

$$e^{At} = e^{Dt}e^{Nt} = e^{2t} \begin{bmatrix} 1-t & 2t & -t \\ 0 & 1 & 0 \\ t & -2t & 1+t \end{bmatrix}.$$

Solution question 5 (5 + 10 + 5 points)

(a)
- The equation $Ty = y$ reads as

$$y(x) = x + \int_0^x \lambda(x-t)y(t)\,dt.$$

  Setting $x = 0$ gives $y(0) = 0$ so the first initial condition is satisfied.
  (1 point)

- Differentiating both sides gives the equation

$$y'(x) = 1 + \int_0^x \lambda y(t)\,dt$$

  Setting $x = 0$ then gives $y'(0) = 1$ so the second initial condition is also satisfied.
  (2 points)

- Differentiating once more gives

$$y''(x) = \lambda y(x)$$

  which shows that the differential equation is satisfied.
  (2 points)

(b) For all $y, z \in C([0, b])$ and $x \in [0, b]$ we have the following inequalities:

$$\begin{aligned} |(Ty)(x) - (Tz)(x)| &= \left| \int_0^x \lambda(x-t)(y(t) - z(t))\,dt \right| \\ &\leq \int_0^x |\lambda| \cdot |y(t) - z(t)| \cdot (x-t)\,dt \\ &\leq |\lambda| \cdot \|y - z\| \int_0^x x - t\,dt, \end{aligned}$$

where we have used that $0 \leq t \leq x$ so that $|x - t| = x - t$.
(5 points)

Moreover, we have

$$\int_0^x x - t\,dt = \left[xt - \tfrac{1}{2}t^2\right]_0^x = \tfrac{1}{2}x^2 \leq \tfrac{1}{2}b^2.$$

(3 points)

Hence, for all $x \in [0, b]$ we have

$$|(Ty)(x) - (Tz)(x)| \leq \tfrac{1}{2}|\lambda|b^2 \cdot \|y - z\|.$$

Taking the supremum over all $x \in [0, b]$ gives the desired inequality.

(2 points)

(c) By definition $y_0(x) = x$ and this is equal to the given sum when $n = 0$. Therefore, the formula is true for $n = 0$.

(1 point)

Now assume that the formula holds for a certain n. Then

$$\begin{aligned}
y_{n+1}(x) &= x + \int_0^x \lambda(x-t)y_n(t)\,dt \\
&= x + \sum_{k=0}^{n} \frac{\lambda^{k+1}}{(2k+1)!}\int_0^x (x-t)t^{2k+1}\,dt \\
&= x + \sum_{k=0}^{n} \frac{\lambda^{k+1}}{(2k+1)!}\left(\frac{x^{2k+3}}{2k+2} - \frac{x^{2k+3}}{2k+3}\right) \\
&= x + \sum_{k=0}^{n} \frac{\lambda^{k+1}x^{2k+3}}{(2k+3)!} \\
&= x + \sum_{k=0}^{n} \frac{\lambda^{k+1}x^{2(k+1)+1}}{(2(k+1)+1)!} \\
&= x + \sum_{k=1}^{n+1} \frac{\lambda^{k}x^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{n+1} \frac{\lambda^{k}x^{2k+1}}{(2k+1)!}
\end{aligned}$$

which shows that the formula also holds for $n + 1$. By induction the formula holds for all $n \in \mathbb{N} \cup \{0\}$.

(4 points)

Solution question 6 (2 + 3 + 5 points)

(a) The equation can be written as $Lu = f$ where

$$Lu = x^2u'' - 2xu' + 2u \quad \text{and} \quad f(x) = x^3 \sin x.$$

The equation is called linear because L is a linear transformation: for all scalars a, b and twice differentiable functions u, v we have that $L(au + bv) = aLu + bLv$. An alternative answer is that the equation is linear because the solutions of the homogeneous equation $Lu = 0$ form a linear space.

(2 points)

(b) Substituting $u = x^\lambda$ into the homogeneous equation $Lu = 0$ gives the following characteristic equation:

$$\lambda(\lambda - 1) - 2\lambda + 2 = 0 \quad \Leftrightarrow \quad (\lambda - 1)(\lambda - 2) = 0.$$

Therefore, the general solution of the homogeneous equation is $u = ax + bx^2$.
(3 points)

(c) We have

$$\begin{aligned} u &= Ax^m \sin x \\ u' &= mAx^{m-1} \sin x + Ax^m \cos x \\ u'' &= m(m-1)Ax^{m-2} \sin x + 2mAx^{m-1} \cos x - Ax^m \sin x \end{aligned}$$

(3 points)
Substitution into the equation $Lu = f$ gives

$$2(m-1)Ax^{m+1} \cos x + \left[(m^2 - 3m + 2)Ax^m - Ax^{m+2}\right] \sin x = x^3 \sin x.$$

Obviously, this equation is satisfied with $m = 1$ and $A = -1$. We conclude that $u_p = -x \sin x$ is a particular solution.
(2 points)

Solution question 7 (15 points)

- For $\lambda = 0$ the solution of the differential equation is $u(x) = ax + b$. The boundary conditions imply that $a = 0$ and b is arbitrary. Therefore, $\lambda = 0$ is an eigenvalue. A corresponding eigenfunction is, for example, given by $u(x) = 1$.
(3 points)

- For $\lambda < 0$ we can write $\lambda = -\mu^2$ and then the solution of the differential equation is
$$u(x) = ae^{\mu x} + be^{-\mu x}.$$
The boundary conditions give the equations
$$\begin{bmatrix} \mu & -\mu \\ \mu e^{\mu\pi} & -\mu e^{-\mu\pi} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
Nonzero solutions for a and b only exist when the determinant of the coefficient matrix is zero:
$$\mu^2(e^{\mu\pi} - e^{-\mu\pi}) = 0.$$
The latter equation only holds for $\mu = 0$, but this contradicts that $\lambda < 0$. Therefore, $\lambda < 0$ is not an eigenvalue.
(6 points)

- For $\lambda > 0$ we can write $\lambda = \mu^2$ in which case the solution of the differential equation is
$$u(x) = a\cos(\mu x) + b\sin(\mu x).$$
The boundary conditions give the equations
$$\begin{bmatrix} 0 & \mu \\ -\mu \sin(\mu\pi) & \mu \cos(\mu\pi) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Nontrivial solutions for a and b only exist when the determinant of the coefficient matrix is zero:

$$\mu^2 \sin(\mu\pi) = 0.$$

The latter equation only holds when $\mu \in \mathbb{Z}$. Note that $\mu = 0$ contradicts the assumption that $\lambda > 0$. Hence, we obtain the eigenvalues $\lambda_n = n^2$ for $n = 1, 2, 3, \ldots$ The corresponding eigenfunctions are, for example, $u_n(x) = \cos(nx)$.

(6 points)